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Construction of Modified Third-Order Upwind Schemes for Stretched Meshes

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Some upwind schemes on stretched meshes are introduced. Our concept for numerical schemes is based on two relations: 1) a polynomial and the derivatives, and 2) a polynomial and the integrations. In this paper, the Taylor series expansion of a function q about a certain point and the Lagrangian interpolation formula are utilized to construct the upwind scheme on a physical space. Three results in two dimensions are shown: a motion of viscous vortex with a uniform background flow, a driven cavity flow, and a flow past a circular cylinder. The method presented produces smooth and realistic results for computations with stretched meshes.

Introduction

SEVERAL third-order upwind schemes have been developed to simulate a high Reynolds number flow, especially in an incompressible flowfield. Many computations have shown that third-order upwinding is reliable in investigating a quantitative feature of the flowfield. However, if these schemes are applied to a computation with a skewed grid system, some types of spurious oscillations are observed in the computed flowfields.

The concept of a numerical transformation from Cartesian coordinates to a general curvilinear system has played an important role in the progression of computational fluid dynamics (CFD). Construction of numerical schemes and the estimation of numerical errors have been discussed first on a computational space and then interpreted to a physical space. Several investigations have been performed on the behavior of truncation errors in transforming the physical space.[1,2] All of these have concluded that sound solutions are obtained by an adequate grid system and that numerical oscillations are caused by a skew cell. In the practical computations, it is difficult to generate a smooth grid system in a whole space because grid generation is restricted by the number of grid points and numerical algorithms. To carry out accurate numerical simulations, two ideas are introduced: the development of the technique for reducing the skewness and the construction of a robust numerical scheme for the nonuniform grid system. In our investigation, the latter is chosen because an independency of the grid system on the numerical solution is the most important matter in CFD.

Our final purpose is to develop upwind schemes for solving a system of equations about a fluid flow on a nonuniform grid system, which prevents numerical instability from affecting computational results and preserves the third-order accuracy, and to inquire into the geometrical behavior of the local solution of governing equations. To achieve this goal, we use the idea of approximate theory and its geometrical interpretation. In this paper, as a building block for a subsequent paper, a basic idea of constructing the modified upwind scheme for one-dimensional advection-diffusion equation is described. The obtained scheme is extended to the multidimensions by the alternating-direction implicit (ADI) method and applied to solving the incompressible Navier-Stokes equations. Although the nonuniform grid system in multidimensions is restricted to the stretched meshes in each direction, it is useful to investigate the numerical instability caused by the stretched grid systems that have often appeared in the cases of the composite or the multiblock grid systems for the Navier-Stokes computations.

Computational Approach

A basic idea of constructing the modified upwind scheme is described for one-dimensional advection-diffusion equation:

$$\frac{\partial q}{\partial t} + u\,\frac{\partial q}{\partial x} = \nu\,\frac{\partial^2 q}{\partial x^2} \tag{1}$$

Our concept for the numerical scheme is based on two relations: 1) a polynomial and the derivatives, and 2) a polynomial and the integrations. At first, we consider the flowfield without discontinuities. In this field, it has been found that the relation between a polynomial and the Taylor series expansion in the physical space is very useful. The Taylor series expansion of a function q about a certain point is represented by the following polynomial:

$$q = \sum_{n=0}^{m} a_n \cdot x^n + \mathcal{O}(x^{m+1}), \qquad a_n = \frac{1}{n!}\cdot\frac{\partial^n q}{\partial x^n} \tag{2}$$

In our analysis, this formulation becomes the basis of construction.

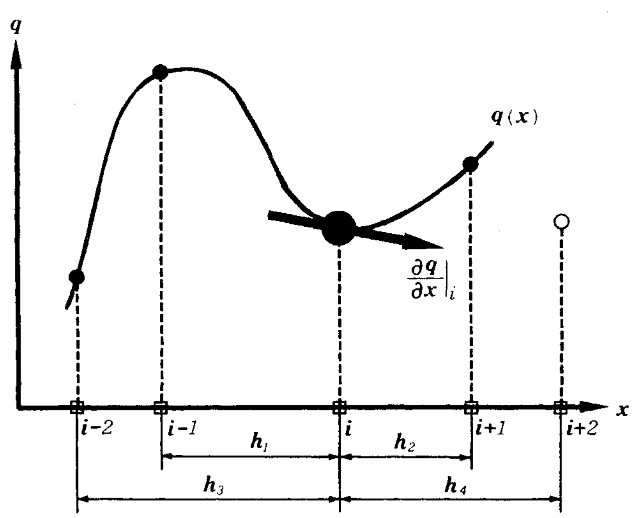

Fig. 1 Four-point system in the upwind direction.

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Then we have proposed the third-order upwind scheme according to Eq. (2). Since a coefficient of the polynomial is expressed by a derivative of q, the derivative with desired precision is available to be determined by using an adequate number of discrete points. To calculate the polynomial, the Lagrangian interpolation formula is used as the explicit method. The m degree polynomial with $m+1$ points is defined uniquely by this interpolation. That polynomial is

$$q = \sum_{n=0}^{m} q_n \cdot \frac{\pi(x)}{(x - x_n)\cdot(\partial\pi/\partial x)_{x=x_n}} \tag{3a}$$

where

$$\pi(x) = \prod_{n=0}^{m}(x - x_n) \tag{3b}$$

For the purpose of demonstrating our scheme, a stencil consisting of five points that are placed at $(x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2})$ is used as shown in Fig. 1. Several notations are introduced for the explanation:

$$h_1 = x_i - x_{i-1}, \qquad h_2 = x_{i+1} - x_i$$

$$h_3 = x_i - x_{i-2}, \qquad h_4 = x_{i+2} - x_i$$

$$r_1 = \frac{h_1}{h_2}, \quad r_3 = \frac{h_3}{h_2}, \quad r_4 = \frac{h_4}{h_2}$$

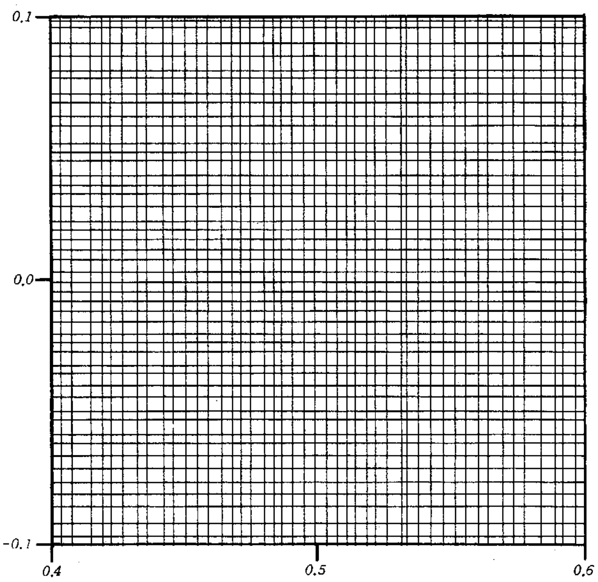

Fig. 2a Grid system for simulation of the movement of viscous vortex (enlarged around a center of vortex at $t = 0.6$).

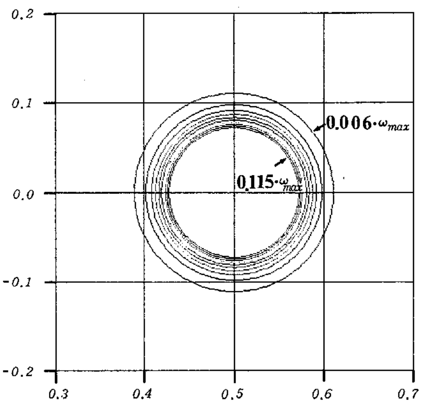

Fig. 2b Vorticity contours at $t = 0.6$ in the case of the exact solution ($Re = 1000$).

To evaluate the third-order upwind scheme, the upwind biased third-degree polynomial with four points is utilized (see Fig. 1). If $u > 0$, then the first derivative at $x = x_i$ is

$$u\left.\frac{\partial q}{\partial x}\right|_{3\mathrm{rd}+} = u \cdot \frac{1}{h_2}\cdot(\alpha_{1+}\cdot q_{i-2} + \beta_{1+}\cdot q_{i-1} + \gamma_{1+}\cdot q_i + \delta_{1+}\cdot q_{i+1}) \tag{4a}$$

where

$$\alpha_{1+} = \frac{r_1}{r_3\cdot(r_3 - r_1)\cdot(r_3+1)}, \quad \beta_{1+} = \frac{-r_3}{r_1\cdot(r_1+1)\cdot(r_3-r_1)}$$

$$\delta_{1+} = \frac{r_3\cdot r_1}{(r_3+1)\cdot(r_1+1)}, \quad \gamma_{1+} = -(\alpha_{1+} + \beta_{1+} + \delta_{1+})$$

If $u < 0$,

$$u\cdot\left.\frac{\partial q}{\partial x}\right|_{3\mathrm{rd}-} = u\cdot\frac{1}{h_2}\cdot(\beta_{1-}\cdot q_{i-1} + \gamma_{1-}\cdot q_i + \delta_{1-}\cdot q_{i+1} + \epsilon_{1-}\cdot q_{i+2}) \tag{4b}$$

where

$$\beta_{1-} = \frac{-r_4}{(r_1+r_4)\cdot(r_1+1)\cdot r_1}, \quad \delta_{1-} = \frac{r_1\cdot r_4}{(r_1+1)\cdot(r_4-1)}$$

$$\epsilon_{1-} = \frac{-r_1}{(r_1+r_4)\cdot(r_4-1)\cdot r_4}, \quad \gamma_{1-} = -(\beta_{1-} + \delta_{1-} + \epsilon_{1-})$$

We define the previous formulation as a pure third-order upwind scheme and use this as the kernel of modified upwind schemes. However, for the computation of high Reynolds number flowfields, the pure upwind scheme induces unstable results. Therefore, a fourth-order dissipation term is added in order to stabilize the scheme. The fourth-order dissipation term is calculated by the fourth-degree polynomial with five

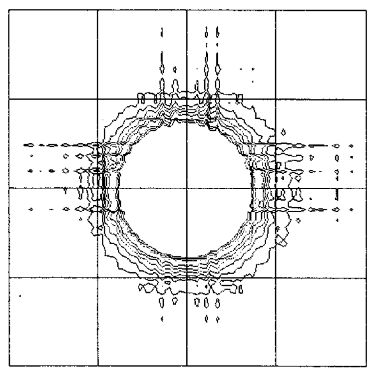

Fig. 2c Solution that corresponds to Fig. 2b is computed by the conventional method.[4-6]

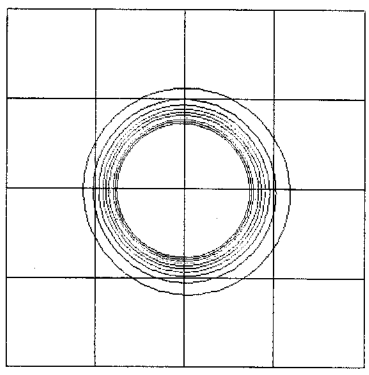

Fig. 2d Present result; $\alpha = 3$.

points. The fourth derivative at $x = x_i$ makes the dissipation term. Thus the derivative is

$$\frac{\partial^4 q}{\partial x^4} = \frac{1}{h_2^4} \cdot (\alpha_4 \cdot q_{i-2} + \beta_4 \cdot q_{i-1} + \gamma_4 \cdot q_i + \delta_4 \cdot q_{i+1} + \epsilon_4 \cdot q_{i+2}) \tag{5}$$

where

$$\alpha_4 = \frac{-24}{(r_1 - r_3) \cdot (r_3 + r_4) \cdot (r_3 + 1) \cdot r_3}$$

$$\beta_4 = \frac{24}{(r_1 - r_3) \cdot (r_1 + r_4) \cdot (r_1 + 1) \cdot r_1}$$

$$\delta_4 = \frac{-24}{(r_1 + 1) \cdot (r_3 + 1) \cdot (r_4 - 1)}$$

$$\epsilon_4 = \frac{24}{(r_1 + r_4) \cdot (r_3 + r_4) \cdot (r_4 - 1) \cdot r_4}$$

$$\gamma_4 = -(\alpha_4 + \beta_4 + \delta_4 + \epsilon_4)$$

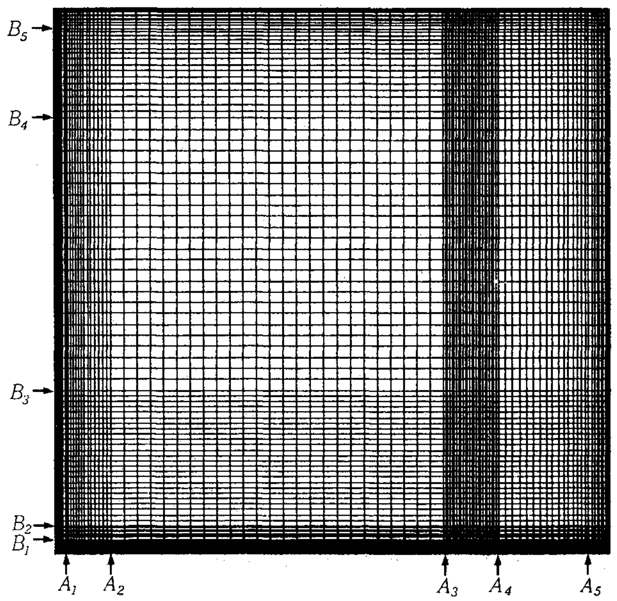

Fig. 3a Grid system for problem of a driven cavity flow.

The so-called modified third-order upwind scheme is represented by blending the fourth-order central difference of the first derivative and the fourth-order dissipation term. On a uniform grid system, the scheme is expressed by

$$u\left.\frac{\partial q}{\partial x}\right|_{3\text{rd}} = u\,\frac{q_{i-2} - 8(q_{i-1} - q_{i+1}) - q_{i+2}}{12\Delta h} + \alpha \cdot |u|\,\frac{q_{i-2} - 4q_{i-1} + 6q_i - 4q_{i+1} + q_{i+2}}{12\Delta h} \tag{6}$$

where α is the blending parameter and Δh represents the grid spacing. The right-hand side of Eq. (6) is rewritten as

$$u\left.\frac{\partial q}{\partial x}\right|_{3\text{rd}} = u\,\frac{\partial q}{\partial x} + \alpha \cdot |u|\,\frac{\Delta h^3}{12}\,\frac{\partial^4 q}{\partial x^4} \tag{7}$$

If α takes 1, the scheme reduces to the UTOPIA scheme proposed by Leonard,[3] and we have defined this scheme as the pure upwind scheme. The scheme in the case of $\alpha = 3$ has been utilized by Kawamura et al.,[4] Shirayama and Kuwahara,[5] and Shirayama.[6] Note that several authors[3-6] have used the fourth-degree polynomial to evaluate the first derivative and the fourth derivative in the case of uniform spacing. In the case of a stretched grid system, the third-order upwind schemes cannot be constructed by using the fourth-degree polynomial, i.e., they cannot be made by blending the fourth-order central difference and the fourth-order dissipation term. The scheme is derived by adding the fourth-order dissipation term to the pure third-order upwind scheme. The reason is explained for the case of the first-order upwind scheme on a stretched grid system. The construction of the first-order upwind scheme is interpreted as blending the second-order central difference and the second-order dissipation term (diffusion term). The formulation is represented by

$$u\left.\frac{\partial q}{\partial x}\right|_{1\text{st}} = u\,\frac{\partial q}{\partial x} - |u|\,\frac{\Delta h}{2}\,\frac{\partial^2 q}{\partial x^2} \tag{8a}$$

The first derivative and the second derivative are evaluated by the second-degree polynomial. The discretized form of Eq. (8a) is

$$u\left.\frac{\partial q}{\partial x}\right|_{1\text{st}} = u\,\frac{1}{h_2}\left[-\frac{1}{r_1 \cdot (r_1 + 1)}\,q_{i-1} - \left(1 - \frac{1}{r_1}\right) \cdot q_i + \frac{r_1}{(r_1 + 1)}\,q_{i+1}\right] - |u|\,\frac{\Delta h}{2}\,\frac{1}{h_2^2}\left[\frac{2}{r_1 \cdot (r_1 + 1)}\,q_{i-1} - \frac{2}{r_1}\,q_i + \frac{2}{(r_1 + 1)}\,q_{i+1}\right] \tag{8b}$$

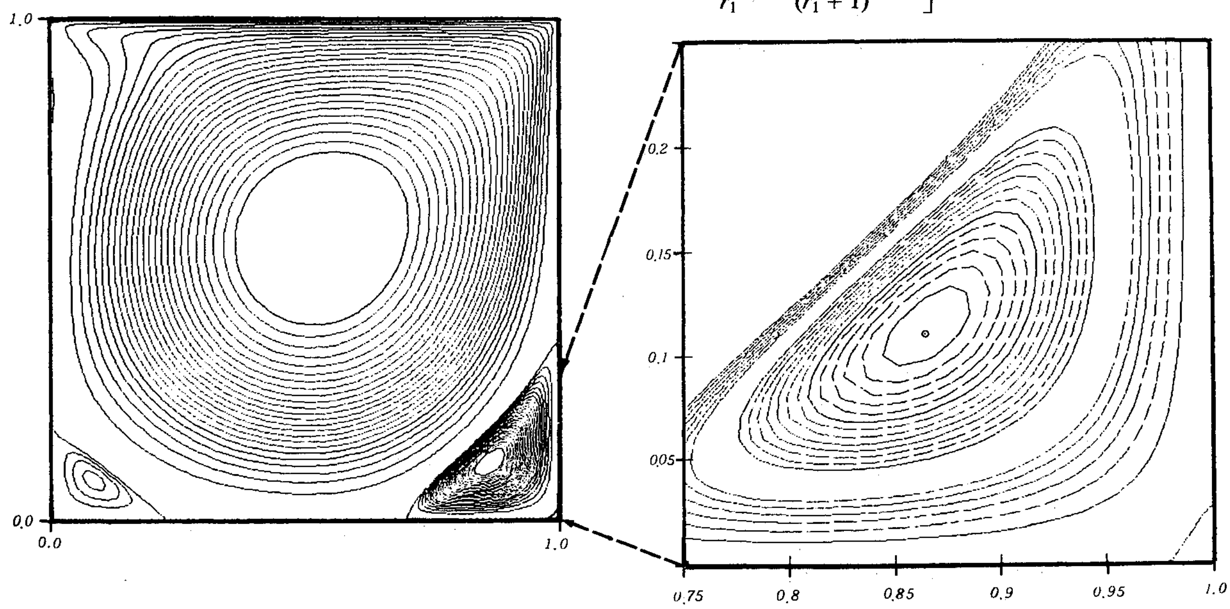

Fig. 3b Streamlines at *Re* = 1000.

Table 1 Comparison between solutions in the case of a driven cavity flow

	Primary vortex, ψ	Location (x,y)	Secondary vortex, bottom right, ψ	Location (x,y)
		$Re = 1000$		
Ghia et al.[7]	− 0.1179	0.5313, 0.5625	0.175E-2	0.8594, 0.1094
Shreiber and Keller[8]	− 0.1160	0.5286, 0.5643	0.170E-2	0.8643, 0.1071
Bruneau et al.[9]	− 0.1193	0.5313, 0.5664	0.174E-2	0.8633, 0.1133
Present	− 0.1147	0.5296, 0.5640	0.163E-2	0.8646, 0.1100
		$Re = 5000$		
Ghia et al.[7]	− 0.1190	0.5117, 0.5352	0.308E-2	0.8086, 0.0742
Bruneau et al.[9]	− 0.1240	0.5156, 0.5352	0.306E-2	0.8086, 0.0742
Present	− 0.1170	0.5100, 0.5360	0.305E-2	0.8140, 0.0780

Let Δh be h_2. Then if $u > 0$,

$$u\left.\frac{\partial q}{\partial x}\right|_{1st} = u\,\frac{q_i - q_{i+1}}{h_1} + u\cdot(h_1 - h_2)\cdot\frac{\partial^2 q}{\partial x^2} \tag{9}$$

The first term in the right-hand side of Eq. (9) denotes the exact form of the first-order upwind scheme on the stretched grid system. Therefore, the derivation from the second-degree polynomial leads to a numerical diffusion error for which the sign depends on $(h_1 - h_2)$. In the case of $u < 0$, a similar result is obtained. The upwind scheme has to be made by adding the dissipation term to the pure upwind scheme.

Here we describe the error caused by a coordinate transformation. In the recent computations, Eq. (7) is utilized in a generalized curvilinear system.[4-6] If a spatial variation of metric coefficients is restricted in the one-dimensional directions, the error caused by the coordinate transformation can be discussed in one dimension. The expression of the advection term is

$$u\,\frac{\partial q}{\partial x} = U\,\frac{\partial q}{\partial \xi} \tag{10}$$

where U is the ξ component of the contravariant velocity vector and given by

$$U = u\,\frac{1}{(\partial x/\partial \xi)}$$

The third-order upwind schemes in the computational space are expressed by

$$U\left.\frac{\partial q}{\partial \xi}\right|_{3rd} = U\,\frac{\partial q}{\partial \xi} + \alpha\cdot|U|\,\frac{\Delta\xi^3}{12}\,\frac{\partial^4 q}{\partial \xi^4} \tag{11}$$

where $\Delta\xi$ is the grid spacing. Since the transformed region is discretized by the uniform grid, the third-order upwind schemes can be constructed by blending the fourth-order central difference and the fourth-order dissipation term. The first term of the right-hand side of Eq. (11) is discretized by the fourth-order central difference. The considerable error caused by the coordinate transformation is a truncation error. According to the work of Thompson et al.,[1] we introduce the following expression (with $\Delta\xi = 1$):

$$\frac{\partial q}{\partial x} = \frac{1}{(\partial x/\partial \xi)}\cdot(q_{i-2} - 4q_{i-1} + 6q_i - 4q_{i+1} - q_{i+2}) + T_1 \tag{12}$$

where T_1 is the truncation error and given by

$$T_1 = -\frac{1}{120}\,\frac{1}{(\partial x/\partial \xi)}\,\frac{\partial^5 q}{\partial \xi^5}\cdots$$

The truncation error T_1 is expressed by the x derivatives as follows:

$$\begin{aligned} T_1 = &-\frac{1}{120}\,\frac{(\partial^5 x/\partial \xi^5)}{(\partial x/\partial \xi)}\,\frac{\partial q}{\partial x} \\ &-\left[\frac{1}{24}\,\frac{\partial^4 x}{\partial \xi^4} + \frac{1}{12}\,\frac{(\partial^2 x/\partial \xi^2)(\partial^3 x/\partial \xi^3)}{(\partial x/\partial \xi)}\right]\frac{\partial^2 q}{\partial x^2} \\ &-\left[\frac{1}{8}\left(\frac{\partial^2 x}{\partial \xi^2}\right)^2 + \frac{1}{12}\,\frac{\partial^3 x}{\partial \xi^3}\,\frac{\partial x}{\partial \xi}\right]\frac{\partial^3 q}{\partial x^3} \\ &-\frac{1}{12}\left(\frac{\partial x}{\partial \xi}\right)^2\frac{\partial^2 x}{\partial \xi^2}\,\frac{\partial^4 q}{\partial x^4} - \frac{1}{120}\left(\frac{\partial x}{\partial \xi}\right)^4\frac{\partial^5 q}{\partial x^5} \end{aligned} \tag{13}$$

The first four terms arise from nonuniform spacing, whereas the last term occurs with uniform spacing. In this case, the truncation error goes to zero in proportion to $1/N^4$ (N is the number of grid points in the one-dimensional direction). However, from the standpoint of the numerical instability, the third term is important. That is, in the higher order upwind scheme, the third derivative generates the numerical instability. In the stretched meshes, this term is not negligible because $(\partial/\partial\xi)(\partial x/\partial\xi)$ increases. Next we show that this term arises from the fourth-order dissipation term of Eq. (11). The fourth-order dissipation term is added to eliminate the numerical instability. This is expressed by the x derivatives as follows:

$$\begin{aligned} \alpha\cdot|U|\,\frac{1}{12}\,\frac{\partial^4 q}{\partial \xi^4} = \alpha\cdot|U|\,\frac{1}{12}&\left\{\frac{(\partial^4 x/\partial \xi^4)}{(\partial x/\partial \xi)}\,\frac{\partial q}{\partial x}\right. \\ &+\left[4\,\frac{\partial^3 x}{\partial \xi^3} + 3\,\frac{(\partial^2 x/\partial \xi^2)^2}{(\partial x/\partial \xi)}\right]\frac{\partial^2 q}{\partial x^2} \\ &\left.+6\,\frac{\partial^2 x}{\partial \xi^2}\,\frac{\partial x}{\partial \xi}\,\frac{\partial^3 q}{\partial x^3} + \left(\frac{\partial x}{\partial \xi}\right)^3\frac{\partial^4 q}{\partial x^4}\right\} \end{aligned} \tag{14}$$

From the previous discussion, it is shown that the coordinate transformation must be performed by using a smoothed grid system. This makes it difficult to generate the grid system in the case of the Navier-Stokes computations. In this way, the modified third-order upwind schemes on the stretched grid systems are introduced and developed. The form is if $u > 0$, then

$$\begin{aligned} u\left.\frac{\partial q}{\partial x}\right|_{3rd} = &\; u\cdot\frac{1}{h_2}\cdot(\alpha_{1+}\cdot q_{i-2} + \beta_{1+}\cdot q_{i-1} + \gamma_{1+}\cdot q_i \\ &+\delta_{1+}\cdot q_{i+1}) + \alpha\cdot u\,\frac{\Delta h^3}{12}\,\frac{1}{h_2^4}\cdot(\alpha_4\cdot q_{i-2} + \beta_4\cdot q_{i-1} \\ &+\gamma_4\cdot q_i + \delta_4\cdot q_{i+1} + \epsilon_4\cdot q_{i+2}) \end{aligned} \tag{15a}$$

and if $u < 0$, then

$$u\left.\frac{\partial q}{\partial x}\right|_{3rd} = u \cdot \frac{1}{h_2} \cdot (\beta_{1-} \cdot q_{i-1} + \gamma_{1-} \cdot q_i + \delta_{1-} \cdot q_{i+1}$$

$$+ \epsilon_{1-} \cdot q_{i+2}) - \alpha \cdot u \frac{\Delta h^3}{12} \frac{1}{h_2^4} \cdot (\alpha_4 \cdot q_{i-2} + \beta_4 \cdot q_{i-1}$$

$$+ \gamma_4 \cdot q_i + \delta_4 \cdot q_{i+1} + \epsilon_4 \cdot q_{i+2}) \tag{15b}$$

The diffusion term is estimated by using the second-degree or the fourth-degree polynomial. In the case of the fourth-degree polynomial, the second derivative at $x = x_i$ makes the diffusion term and is given by

$$\frac{\partial^2 q}{\partial x^2} = \frac{1}{h_2^2} \cdot (\alpha_2 \cdot q_{i-2} + \beta_2 \cdot q_{i-1} + \gamma_2 \cdot q_i + \delta_2 \cdot q_{i+1} + \epsilon_2 \cdot q_{i+2}) \tag{16}$$

where

$$\alpha_2 = \frac{2[(r_4 + 1) \cdot r_1 - r_4]}{(r_1 - r_3) \cdot (r_3 + r_4) \cdot (r_3 + 1) \cdot r_3}$$

$$\beta_2 = \frac{-2[(r_4 + 1) \cdot r_3 - r_4]}{(r_1 - r_3) \cdot (r_1 + r_4) \cdot (r_1 + 1) \cdot r_1}$$

$$\delta_2 = \frac{2[(r_3 - r_4) \cdot r_1 - r_3 \cdot r_4]}{(r_1 + 1) \cdot (r_3 + 1) \cdot (r_4 - 1)}$$

$$\epsilon_2 = \frac{-2[(r_3 - 1) \cdot r_1 - r_3]}{(r_1 + r_4) \cdot (r_3 + r_4)(r_4 - 1) \cdot r_4}$$

$$\gamma_2 = -(\alpha_2 + \beta_2 + \delta_2 + \epsilon_2)$$

For a scheme with a higher accuracy, more grid points have to be used. The truncation error of Eq. (2), however, becomes large in proportion to the H^{m+1} [$H = \max(h_i)_{i=1,\ldots,m+1}$] if the stretched grid system is utilized. Many computational results have shown the accuracy of the third-order interpolated schemes.[3-6] We consider the local solution of the Navier-Stokes equations to be sufficiently approximated by the cubic interpolation. This heuristic statement allows us to choose the third-order polynomial.

Results and Discussion

The grid systems are composed of the stretched grid in one dimension. In the multidimensional computations, the Descartes, the cylindrical, and the spherical coordinates are utilized. The computed flowfields are obtained by solving the incompressible Navier-Stokes equations.

Three results in two dimensions are shown: a motion of viscous vortex with a uniform background flow (shown in Fig. 2), a driven cavity flow (presented in Fig. 3 and Table 1), and flow past a circular cylinder (Fig. 5). All computations are performed using a grid system with a distorted portion defined by Ref. 2 or a randomly distributed grid system. The first and third results are compared with the results obtained by the conventional method.[4-6] The conventional method is constructed in the generalized curvilinear coordinates. The advection term is discretized by the third-order upwind scheme [i.e., Eq. (11)].

Motion of Viscous Vortex with a Uniform Background Flow

The governing equation is the vorticity equation:

$$\frac{\partial \omega}{\partial t} + a\frac{\partial \omega}{\partial x} + b\frac{\partial \omega}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right)$$

where a and b denote the component of the velocity of uniform flow. An exact solution of this equation is

$$\omega = \frac{\Omega_0}{4\pi\tau} \exp\left[-\frac{(x - at)^2 + (y - bt)^2}{4\tau}\right]$$

where Ω_0 represents the strength of the initial vorticity and $\tau = t/Re$. The Reynolds number is 1000. The initial condition is set as the exact solution at $t = 0.1$ with $\Omega_0 = 1$. The center of the vortex is located at (0, 0). The uniform velocity is (1, 0). The boundary condition is given by the exact solution. The CFL number is about 0.5. We have performed some computations with different grid systems. The typical grid system is

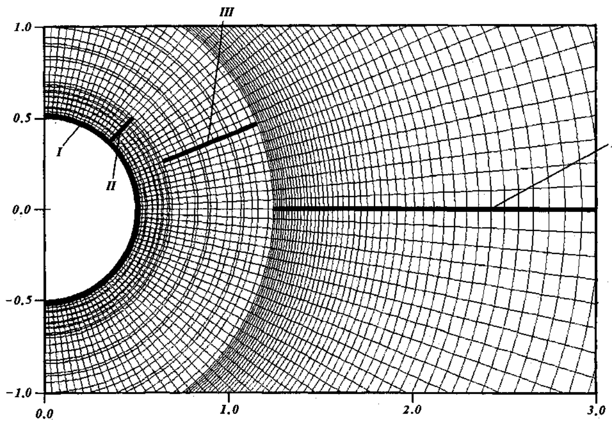

Fig. 4a Grid system for analyzing flow around a circular cylinder. Initial grid is composed of four portions with uniform grid spacing, then grid points in regions II and III are randomly distributed. In region IV, grid points are concentrated on the boundary between regions III and IV.

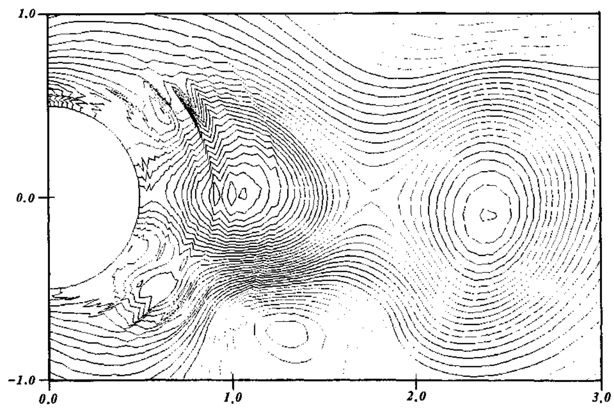

Fig. 4b Pressure contours at $Re = 1000$. The result is computed by the conventional method.[4-6]

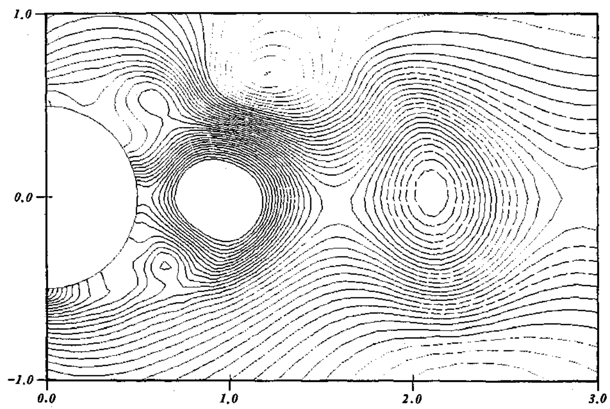

Fig. 4c Present result; $\alpha = 3$.

demonstrated in Fig. 2a. Figure 2a shows the grid system enlarged around a center of vortex at $t = 0.6$. The exact solution is presented in Fig. 2b. Since the grid points are randomly distributed as seen in Fig. 2a, the variations of $\partial x/\partial\xi$, $\partial^2 x/\partial\xi^2$, $\partial^3 x/\partial\xi^3 \ldots$ are not smooth in a whole region. If we use the techniques of the coordinate transformation, it is predicted from Eqs. (13) and (14) that the numerical results are affected by the grid systems. Particularly, the influence of $\partial^2 x/\partial\xi^2$ is not negligible in a whole region. Figure 2c demonstrates the result by the conventional method. In fact, the numerical instabilities are observed in the wide region, and the instabilities cause the artificial vorticities. Even in this simple problem, the numerical solutions depend largely on the mesh stretchings. While the result by the present method (in Fig. 2d) is smooth, it is in good agreement with the exact solution. Also, it has been found by several numerical experiments using the different grid systems that the dependency of the numerical solutions obtained by the present method on the one-dimensional nonuniform grid is small in the case of the linear advection-diffusion problems.

Driven Cavity Flow

Second, we test our method for the nonlinear problem. The problem is a simulation of a driven cavity flow. Since we compute the case that the Reynolds numbers are 1000 and 5000, it is expected that the flowfields become steady states. The grid system is shown in Fig. 3a. The number of grid points is 100×100. The distribution of the grid points is abruptly changed at the locations indicated by the characters A and B in Fig. 3a. It has been reported by many authors in some conferences or workshops that they have failed in the computations using such discontinuous grid spacing. We also could not obtain the solution by using the conventional method in the present case. The failure of the numerical simulation by the conventional method was caused by such stretching of the grid. The spatial variation of $\partial x/\partial\xi$ is very large at the points A and B. This induces the large truncation errors at those points, and the numerical instabilities prevent the computation from being continued.

Therefore, we show the numerical results by the present method only. The flowfield is characterized by several vortices. The numerical solutions have been compared with each other by acquiring the location and the strength of several vortices.[7-9] Table 1 shows the comparison of present results with other results.[7-9] The location and the strength (the value of the stream function at the center of the vortices) are in good agreement with the results by other methods. Streamlines are presented in Fig. 3b. The secondary and the tertiary vortices are clearly shown in this figure. Nevertheless the regions indicated by A and B exist in such vortices, we cannot observe the numerical instability.

Flow Past a Circular Cylinder

Finally, we compute a flow past a circular cylinder for the purpose of testing our method in the case of the nonlinear unsteady flow problem. As shown in Fig. 4a, the grid system consists of four portions. Region I is assigned to the boundary layer of the circular cylinder and is composed of the uniform spacing grid. In regions II and III, a near wake will be captured, and the grid points are randomly distributed. In region IV, the grid points are concentrated on the boundary between regions III and IV and are smoothly distributed. This region IV is added to simulate a far-wake region. Each region is connected smoothly as the conventional method does work. This kind of grid system appears in the computation of the viscous flowfield with the far-wake region, but the present grid system is deformed to emphasize the difference of the results between the conventional and the present methods. The number of grid points is 150×150. The Reynolds number based on the diameter of the cylinder is 1000. The pressure contours are shown in Figs. 4b and 4c. In this Reynolds number, two kinds of vortices characterize the flowfield. One is the primary vortex that forms the Kármán vortex street. The other is the secondary vortex induced by the primary vortex. The secondary vortices are only apparent in the near-wake region (in regions II and III). The vortices are merged into the primary vortices and flow out to the far-wake region. In the near wake region, the pressure distribution is characterized by the lower pressure regions induced by these vortices. The result by the present method (in Fig. 4b) represents this situation well. However, in the result by the conventional method, we can find several lower pressure regions caused by the numerical instability (in Fig. 4c). The stretching of the grid system generates the artificial vortices in the computation of the vortical flowfields as has been seen in the case of the linear advection-diffusion problem, whereas present results may not be affected by the mesh stretching.

Conclusion

We have proposed a concept of constructing an accurate scheme on grid systems with one-dimensional nonuniformities. Our concept for the numerical scheme is based on the relation between the Taylor series expansion of a function q about a certain point and a certain polynomial: the geometrical interpretation of the local solution of the governing equations. It is shown that the upwind scheme is obtained from the first derivative of the upwind biased polynomial and is not acquired by blending the fourth-order central difference and the fourth-order dissipation term on the nonuniform grid spacing in one dimension.

We have computed several flowfields by using this proposed idea. Although the conventional third-order upwind methods which are constructed in the generalized curvilinear systems do not work well on the stretched grid system, the present method produces smooth and realistic results for computations with stretched meshes. If we use the Taylor series expansion in multidimensions, the same idea is utilized in the construction of highly accurate schemes on the nonuniform grid systems in multidimensions. Our further investigation is directed toward this.

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